

APPROXIMATE MOTION SYNTHESIS VIA PARAMETRIC CONSTRAINT MANIFOLD FITTING

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Abstract. In this paper we present a novel dyad dimensional synthesis technique for approximate motion synthesis. The methodology utilizes an analytic representation of the dyad's constraint manifold that is parameterized by its joint variables. Nonlinear optimization techniques are then employed to minimize the distance from the dyad's constraint manifold to a finite number of desired locations of the workpiece. The result is an approximate motion dimensional synthesis technique that is applicable to planar dyads. Here, we specifically address planar RR dyads since these are often found in the kinematic structure of industrial robotic systems and mechanisms. These dyads may be combined serially to form a complex open chain or, when connected back to the fixed link, may be joined so as to form a closed chain; e.g. a platform or mechanism. Finally, we present a numerical design case study which demonstrate the utility of the synthesis technique.

1. Introduction

The constraint manifold of a dyad represents the geometric constraint imposed on the motion of the moving body or workpiece. This geometric constraint on the moving body is a result of the kinematic structure of the dyad; e.g. its length and the location of its fixed and moving axes (i.e. lines). The constraint manifold is an analytical representation of the workspace of the dyad which is parameterized by the dyad's dimensional synthesis variables. Here we derive the constraint manifold of planar RR dyads in the image space of planar displacements and utilize this constraint manifold to perform dyadic dimensional synthesis for approximate rigid body guidance. The derivation of the constraint manifold in the image space involves writing the kinematic constraint equations of the dyad using the components of a dual quaternion. We view these equations as constraint manifolds in the image space of spatial displacements, see Bottema and Roth (1979)

and Larochelle (1994). The result is an analytical representation of the workspace of the dyad which is parameterized by its joint variables. The synthesis goal is to vary the design variables such that all of the prescribed locations are either: (1) in the workspace, or, (2) the workspace comes as close as possible to all of the desired locations. Recall that in general five is the largest number of locations for which an exact solution is possible for the planar RR dyads being discussed here, see Suh and Radcliffe (1978). Previous works discussing constraint manifold fitting for an arbitrary number of locations include Ravani and Roth (1983), Bodduluri and McCarthy (1992), Bodduluri (1990), and Larochelle (1994). All of these works employ *implicit* representations of the dyad constraint manifolds. The constraint manifolds, which are known to be highly nonlinear (McCarthy 1990), are then linearly approximated by tangent hyperplanes by using a simple Taylor series strategy. The distance from the approximating tangent plane to the desired location is then used to formulate an objective function to be minimized. These efforts met with limited success since the constraint manifolds are highly nonlinear and the approximating tangent planes yield poor measures of the distance from the constraint manifold to the desired locations, see Larochelle (1996) and Larochelle (1994). For example, when solving for a spherical four-bar mechanism for 10 desired locations Bodduluri and McCarthy (1992) utilized 120 starting cases of which 38 converged to the solution. In Larochelle (1996) a methodology which does not require linearization of the constraint manifold was reported. The method utilized computer graphics to visually present a projection of the constraint manifold to the designer. The designer then directly manipulated the synthesis variables until the *parameterized* constraint manifold was acceptably near the desired locations. This technique proved to be effective but was tedious to use and results depended heavily upon the designers experience and knowledge. Here, we utilize *parameterized* constraint manifolds and nonlinear optimization to yield a numerical dimensional synthesis technique for approximate motion synthesis which does not require a linear approximation of the constraint manifold.

We proceed by reviewing the image space of planar displacements and deriving the parameterized form of the constraint manifold of the planar RR dyad. We then present the approximate motion synthesis procedure and a detailed numerical example.

2. Image Space of Planar Displacements

First, we review the use of planar quaternions for describing planar rigid-body displacements. Our approach is to view planar displacements as a subgroup of $SE(3)$ and as occurring in the $X - Y$ plane. A general planar

displacement may then be described by a 3×3 orthonormal rotation matrix $[A]$ and a translation vector $\mathbf{d} = [d_x \ d_y \ 1]^T$. Associated with the matrix of rotation $[A]$ is an axis of rotation $\mathbf{s} = [0 \ 0 \ 1]^T$ and a rotation angle θ . Using the translation vector \mathbf{d} and the rotation angle θ , we can represent a planar displacement by the four nonzero components of a dual quaternion \mathbf{q} which is written as, see McCarthy (1990),

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \frac{d_x}{2} \cos \frac{\theta}{2} + \frac{d_y}{2} \sin \frac{\theta}{2} \\ -\frac{d_x}{2} \sin \frac{\theta}{2} + \frac{d_y}{2} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{bmatrix}. \quad (1)$$

We refer to \mathbf{q} as a planar quaternion. Note that the four components of \mathbf{q} satisfy $q_3^2 + q_4^2 - 1 = 0$ and therefore they form a three dimensional algebraic manifold which we denote as *the image space of planar displacements*.

2.1. PLANAR QUATERNION PRODUCT

Given two planar quaternions, \mathbf{g} and \mathbf{h} , their product yields a planar quaternion which represents the composite planar displacement obtained by the successive application of \mathbf{g} and \mathbf{h} . We may write the product of two planar quaternions in the following matrix form, see McCarthy (1990),

$$\mathbf{gh} = G^+ \mathbf{h} = H^- \mathbf{g} \quad (2)$$

where,

$$G^+ = \begin{bmatrix} g_4 & -g_3 & g_2 & g_1 \\ g_3 & g_4 & -g_1 & g_2 \\ 0 & 0 & g_4 & g_3 \\ 0 & 0 & -g_3 & g_4 \end{bmatrix}$$

and,

$$H^- = \begin{bmatrix} h_4 & h_3 & -h_2 & h_1 \\ -h_3 & h_4 & h_1 & h_2 \\ 0 & 0 & h_4 & h_3 \\ 0 & 0 & -h_3 & h_4 \end{bmatrix}.$$

3. Planar RR Constraint Manifold

In this section we derive the parametric form of the constraint manifold of the planar RR dyad. The constraint manifold is derived by expressing

analytically the geometric structure that the joints of the dyad impose on the moving body, see Larochelle (1994), Ge (1990), Bodduluri (1990), and Suh and Radcliffe (1978). Using the image space representation of planar displacements and the geometric constraint equations of the dyad we arrive at constraint equations in the image space that are parameterized by the joint variables of the dyad.

A planar RR dyad of length a is shown in Fig. (1). Let the axis of the fixed joint be specified by the vector \mathbf{u} measured in the fixed reference frame F and let the origin of the moving frame be specified by \mathbf{v} measured in the link frame A . The dimensional synthesis variables of the dyad are \mathbf{u} , \mathbf{v} , and a . We obtain the structure equation in the image space of planar

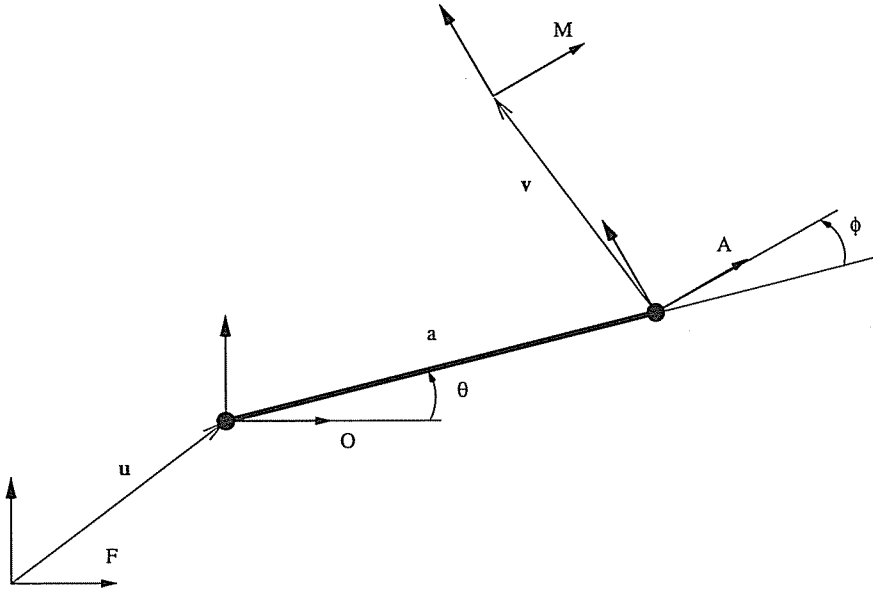


Figure 1. A Planar RR Dyad

displacements by using planar quaternions to represent the displacement \mathbf{D} from F to M ,

$$\mathbf{D} = \mathbf{x}(u_x)\mathbf{y}(u_y)\mathbf{z}(\theta)\mathbf{x}(a)\mathbf{z}(\phi)\mathbf{x}(v_x)\mathbf{y}(v_y) \quad (3)$$

where $\mathbf{x}(\cdot)$, $\mathbf{y}(\cdot)$, and $\mathbf{z}(\cdot)$ are planar quaternion representations of displacements either along or about the X , Y , or Z axes respectively. To take full advantage of the image space representation we now rewrite \mathbf{D} as,

$$\mathbf{D} = \mathbf{g}\mathbf{d}'\mathbf{h} \quad (4)$$

where: $\mathbf{g} = \mathbf{x}(u_x)\mathbf{y}(u_y)$ is the displacement from F to O , $\mathbf{h} = \mathbf{x}(v_x)\mathbf{y}(v_y)$ is the displacement from A to M , and $\mathbf{d}' = \mathbf{z}(\theta)\mathbf{x}(a)\mathbf{z}(\phi)$ is the displacement

along the dyad from O to A. Performing the quaternion multiplications yields,

$$\mathbf{d}' = \begin{bmatrix} \frac{a}{2} \cos \frac{\theta - \phi}{2} \\ \frac{a}{2} \sin \frac{\theta - \phi}{2} \\ \sin \frac{\theta + \phi}{2} \\ \cos \frac{\theta + \phi}{2} \end{bmatrix} \quad (5)$$

$$\mathbf{g} = \begin{bmatrix} \frac{u_x}{2} \\ \frac{u_y}{2} \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{h} = \begin{bmatrix} \frac{v_x}{2} \\ \frac{v_y}{2} \\ 0 \\ 1 \end{bmatrix}. \quad (6)$$

Finally, using Eq. (2) we express Eq. (4) as,

$$\mathbf{D}(\theta, \phi, \mathbf{r}) = \mathbf{c}\mathbf{d}' = G^+(\mathbf{u})H^-(\mathbf{v})\mathbf{d}'(a, \theta, \phi) \quad (7)$$

where $\mathbf{r} = [\mathbf{u}^T \ \mathbf{v}^T \ a]^T$ is the vector of dimensional synthesis variables. In Eq. (7) we have a surface in the image space of planar displacements which is parameterized by the design variables of the dyad. This surface is the constraint manifold of the planar RR dyad. Specifically, for a given fixed pivot \mathbf{u} , a given moving pivot \mathbf{v} , and a crank length a , Eq. (7) yields the constraint manifold of the dyad parameterized by its two joint angles θ and ϕ . Moreover, it is important to note the arrangement of the design variables in Eq. (7). The two joint angles of the dyad have been isolated into the far right hand-side of Eq. (7). This arrangement of the design variables will be exploited later by the approximate motion synthesis technique.

4. Approximate Motion Synthesis

In this section we begin by discussing the metric used to measure the distance between the desired image points and the dyad's constraint manifold. This is followed by a numerical synthesis procedure for designing dyads for approximate motion synthesis.

4.1. THE METRIC

The metric used to measure the distance d between an image point $\mathbf{D}(\theta, \phi, \mathbf{r})$ on a dyad's constraint manifold and an image point \mathbf{q} associated with a desired location of the workpiece is:

$$d(\mathbf{D}(\theta, \phi, \mathbf{r}), \mathbf{q}) = \sqrt{(\mathbf{D}(\theta, \phi, \mathbf{r}) - \mathbf{q})^T (\mathbf{D}(\theta, \phi, \mathbf{r}) - \mathbf{q})}. \quad (8)$$

In order to synthesize dyads which guide the workpiece as near as possible to the desired locations we require an efficient technique for determining the image point $\mathbf{D}(\theta, \phi, \mathbf{r})$ that minimizes d . For a given dyad d_{min} is determined by performing a direct search of a two dimensional fine discretization of the constraint manifold with respect to θ and ϕ . Note that we exploit the separation of variables in generating the discretization of $\mathbf{D}(\theta, \phi, \mathbf{r})$ in Eq. (7) since $G^+(\mathbf{u})H^-(\mathbf{v})$ are constant for a given dyad (i.e. \mathbf{r}).

It is important to note that d is a measure of the *distance* from \mathbf{q} to \mathbf{s} and that even though this metric is useful for designing dyads it, like all other distance metrics, is variant with respect to choice of coordinate system when used for spatial or planar displacements. For further discussions of displacement metrics see Larochelle (1999), Chirikjian (1998), Larochelle and Tse (1998), Gupta (1997), Bobrow and Park (1995), Park (1995), Etzel and McCarthy (1996), Martinez and Duffy (1995), and, Kazerounian and Rastegar (1992).

4.2. THE OPTIMIZATION PROBLEM

Given a finite set of n desired locations the task is to determine the dyad which guides the workpiece through, or as near as possible, to these locations. Our approach is to utilize the metric discussed above to determine the distance from the constraint manifold to each of the n desired locations, sum these distances, and then to employ nonlinear optimization techniques to vary the dimensional synthesis parameters such that the total distance is minimized. The optimization problem then becomes:

$$\text{MINIMIZE: } f(\mathbf{r}) \quad \text{where: } f(\mathbf{r}) = \sum_{i=1}^n d_{min}(\mathbf{r}, \mathbf{q}_i).$$

We utilize the non-linear optimization package ADS by Vanderplaats (1984) with the *variable metric method for unconstrained minimization* by Davidon (1959), and Fletcher and Powell (1963).

5. Case Study

We now present an example of the design of a planar RR dyad for the ten desired locations that were used by Ravani and Roth (1983) to demonstrate their synthesis procedure, see Tbl. (1). The optimal dyad reported by Ravani and Roth was: $\mathbf{u} = [14.00 \quad -0.12]^T$, $\mathbf{v} = [-9.00 \quad 1.00]^T$, and $a = 8.31$. This dyad has a distance sum of $1.03E-2$. The optimal dyad determined here is: $\mathbf{u} = [14.98 \quad -2.08]^T$, $\mathbf{v} = [-11.22 \quad 4.62]^T$, and $a = 6.45$. The distance to each of the desired locations is listed in Tbl. (1) and the distance sum is $3.62E-3$. Note that the distance for the synthesis technique presented here is more than 5 times smaller than that for the dyad determined by the constraint manifold linearization technique of Ravani and Roth. More-

over, our implementation of the methodology of Ravani and Roth (1983) required more than 50 random initial guesses of the solution to have one converge to the optimal dyad they reported while the technique presented here required only one random initialization to converge to the reported solution.

Pos. #	x	y	θ	Distance
1	0.0	0.0	40.0	$1.10E-4$
2	4.5	4.0	20.0	$5.55E-4$
3	8.5	8.0	0.0	$1.08E-3$
4	13.0	11.5	-30.0	$1.50E-4$
5	13.0	12.5	-35.0	$3.27E-5$
6	9.5	14.0	-35.0	$4.23E-4$
7	5.0	13.5	-30.0	$1.22E-5$
8	1.0	10.5	-15.0	$8.09E-4$
9	-1.0	6.5	0.0	$3.93E-4$
10	-1.5	3.0	20.0	$4.76E-5$

TABLE 1. Planar Locations and Distances

6. Conclusions

In this paper we have presented a novel dyad dimensional synthesis technique for approximate motion synthesis for a finite number of desired locations of a workpiece. The methodology utilizes an analytic representation of the dyad's constraint manifold that is parameterized by its joint angles. Nonlinear optimization techniques are then employed to minimize the distance from the dyad's constraint manifold to a finite number of desired locations of the workpiece. It is important to note that the technique presented utilizes a direct search of the discretization of the constraint manifold and thereby avoids the difficulty of previous techniques which required linearization of the constraint manifold. The result is an approximate motion dimensional synthesis technique that is applicable to the design of planar dyads. Currently, work is ongoing to extended the technique to the design of spherical and spatial dyads.

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